Math 128A: Homework 6

Due: July 26

1. Let f and g be Lipschitz on a bounded interval $[a, b]$.
   (a) Prove that both $f + g$ and fg are Lipschitz on $[a, b]$.
   (b) Do the results in (a) still hold if the interval is unbounded? Prove or give a counter-example.

2. Recall that the error formula for an n-point Gauss quadrature scheme is
$$\frac{f^{(2n)}(c)}{(2n)!} \int_{-1}^{1} (p_n(x))^2 \, dx$$
where p_n is the nth monic Legendre polynomial.
   (a) Given that
$$\int_{-1}^{1} (p_n(x))^2 \, dx = 2 \prod_{i=1}^{n} \frac{i^2}{4i^2 - 1},$$
find the smallest n for which the n-point Gauss quadrature scheme will approximate $I = \int_2^3 \frac{1}{x} \, dx$ to within 10^{-6}.
   (b) Using the function `gauss_grid`, generate the nodes and weights for the n found in (a), approximate I and find the actual error.

3. Consider the initial value problem
$$y'(t) = f(t, y) \text{ on } a \le t \le b, \quad y(a) = y_a. \tag{1}$$
The Backward Euler method for (1) is
$$u_0 = y_a, \quad u_{i+1} \quad = \quad u_i + hf(t_{i+1}, u_{i+1}), \quad 0 \le i < n \tag{2}$$
   (a) Assuming that $|y''(t)| \le M$ for some $M > 0$, show that the local truncation error for (2) is $O(h^2)$.
   (b) Assuming that f is Lipschitz in y, prove that (2) is a convergent scheme and find the order of convergence. (Follow the proof of Forward Euler and find a relation between errors at successive time-steps; use $(1 - hL)^{-1} \le e^{\frac{hL}{1-hL}}$ to find a measure of the total error)

4. Consider the initial value problem $y'(t) = -6y$ on $0 \le t \le 4$ with $y(0) = 1$.

(a) Find the exact value of $y(4)$.

(b) Using $h = 0.4$, analytically find the Forward Euler approximation to $y(4)$. Does this contradict the convergence theorem for this method?

(c) Using the same value of h, find the Backward Euler approximation to $y(4)$.

5. Consider the initial value problem $y'(t) = t\sin(y)$ on $0 \le t \le 2$, $y(0) = \pi/2$.

   (a) show that it has a unique solution.

   (b) by finding the exact solution or otherwise, find an upper bound for $|y''(t)|$.

   (c) find h such that solving it with Forward Euler with a step-size of h will guarantee an error of less than 10^{-6}.

   (d) approximate $y(b)$ by solving it with Forward Euler using the step-size found in (c).

6. (a) Solve the initial value problem

$$y'(t) = \sin(t) - \frac{y}{2} \text{ on } 0 \le t \le 8, \quad y(0) = 4$$

   using (i) the third-order Taylor method (ii) Heun's method (iii) the midpoint method and (iv) Ralston's method with $h = 0.25$. Plot the solutions on the same axes.

   (b) As method (i) is third-order, we can safely assume that it is the most accurate. Rank the three Runge-Kutta methods ((ii), (iii), (iv)) from most-to-least accurate by comparing their solutions with that from method (i).

7. Consider the explicit two-stage Runge-Kutta method

$$\begin{array}{c|cc} 0 & 0 & \\ a & a & \\ \hline & b_1 & b_2 \end{array}$$

In class, we derived the equations for a, b_1 and b_2 that, if satisfied, ensured that the local truncation error (LTE) of the method was $O(h^3)$. This problem arrives at the same result in a much more elegant way.

   (a) Suppose that the above method indeed has an LTE of $O(h^3)$. By considering $y'(t) = f(t)$ and $h = 1$, show that $x_1 = 0$, $x_2 = a$ and $w_1 = b_1$, $w_2 = b_2$ are the points and weights of some numerical integration rule on $[0, 1]$ which is exact for all polynomials of degree 1 and lower. (Hint: what is the coefficient of h^3 in the LTE expression?)

   (b) Use (a) to show that any method of the form above with an LTE of $O(h^3)$ must satisfy

$$b_1 + b_2 = 1, \quad ab_2 = \frac{1}{2}.$$